This new method would seem to pose quite a difficult requirement: that of measuring fluctuating static pressure in a supersonic flow. However, the difficulty is mitigated considerably because pressure fluctuations are usually smaller than fluctuations in other variables. Thus, the pressure terms of Eqs. (11-16) represent only a correction to the results obtained by setting $p'/p = 0$. Sufficient accuracy might be obtained even if $\langle p'/p \rangle$ is merely estimated rather than measured exactly. Such estimates might be obtained from wall pressure measurements or a microphone static pressure probe—techniques that are both simple and relatively inexpensive to implement. Even if $\langle p'/p \rangle$ is completely unknown, the direction of the error incurred by neglecting it is apparent by examining Eqs. (11-16) and (26).

Conclusions

A new method of analyzing hot-wire data in supersonic turbulence to take into account the effects of the sound field has been presented. It has the advantage of producing increased accuracy in the fluctuations of flow variables calculated from hot-wire data, and it requires only a moderately accurate estimate of static pressure fluctuations in addition to the usual hot-wire measurements. The method also shows how neglect of pressure fluctuations affects hot-wire data analysis, and shows the probable direction of error.

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Numerical Verification of Design Sensitivity Analysis

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I. Introduction

CONSIDERABLE progress has been made in developing analytical methods for design sensitivity analysis of structural and mechanical systems.[1,2] Numerical accuracy of sensitivity vectors effects the search direction and gradient of the Lagrangian that are used in optimization algorithms. Most sensitivity expressions require numerical implementation of several steps before the final sensitivity vectors are obtained. This is particularly true for problems involving transient dynamic response and shape optimization. Therefore, it is important to verify numerical implementation and calculations based on the analytical sensitivity expressions before optimization of the problem is carried out. This paper describes three numerical procedures for verification of the sensitivity calculations. An example from optimal control literature is used to demonstrate the procedures and make recommendations for practical applications.

II. Verification Procedures

For the purpose of describing the procedures, let $\psi(\boldsymbol{b})$ be a functional whose analytical sensitivity needs to be verified where $\boldsymbol{b}$ is a design variable vector. Let $\boldsymbol{g}(\boldsymbol{b})$ be the gradient (sensitivity vector) of $\psi(\boldsymbol{b})$, which is assumed differentiable.

First Variation Approach

The first procedure is based on calculating first variation of the functional $\psi(\boldsymbol{b})$ and its comparison with the finite-difference evaluation. Consider a nominal design $\boldsymbol{b}$ and its neighboring design $\boldsymbol{b} + \tau\Delta\boldsymbol{b}$ described by an arbitrary design variation $\Delta\boldsymbol{b}$ and a small parameter τ. The first variation of the functional ψ, written as $\psi'(\boldsymbol{b},\Delta\boldsymbol{b})$, is defined as

$$\psi'(\boldsymbol{b},\Delta\boldsymbol{b}) = \frac{\mathrm{d}}{\mathrm{d}\tau}\psi(\boldsymbol{b} + \tau\Delta\boldsymbol{b})|_{\tau=0} = \boldsymbol{g}^T\Delta\boldsymbol{b} \tag{1}$$

where $\boldsymbol{g}$ is the derivative vector obtained from analytical design sensitivity calculations. Also, the difference between $\psi(\boldsymbol{b})$ and $\psi(\boldsymbol{b} + \Delta\boldsymbol{b})$ is given as $\Delta\psi = \psi(\boldsymbol{b} + \Delta\boldsymbol{b}) - \psi(\boldsymbol{b})$. Then, closeness of the number $(\psi'/\Delta\psi) \times 100$ to 100 can be used as a measure of accuracy of the derivative. This procedure globally verifies the gradient calculation of the functional ψ.

Relative Error for the Absolute Maximum Component

An alternate way is to locally verify sensitivity analysis by comparing each of the sensitivity coefficients obtained by finite difference (FDM) and the analytical methods. The procedure is to simply perturb one design variable at a time and use finite differences to approximate each sensitivity coefficient of the functional ψ as

$$(g_i)_{\text{FDM}} = \frac{\mathrm{d}\psi}{\mathrm{d}b_i} \approx \frac{[\psi(\boldsymbol{b} + \delta\boldsymbol{e}^i) - \psi(\boldsymbol{b})]}{\delta} \tag{2}$$

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where vector e^i has unit element in the ith position and zeros elsewhere, and δ is a small perturbation in b_i. This component by component comparison sometimes encounters difficulty, especially when there is a large difference (absolute) between the largest and the smallest components. The smallest component (absolute) may be dominated by the numerical noise. Therefore, a large error in the smallest component does not imply inaccurate gradient. In this case, relative error for the absolute maximum component can be used to locally compare accuracy of the sensitivity coefficients.

The relative error (RE) is defined as

$$\mathrm{RE} = \frac{|(g_i)_{\mathrm{FDM}} - (g_i)_{\mathrm{Anal}}|}{|(g_i)_{\mathrm{FDM}}|} \tag{3}$$

where i corresponds to the absolute maximum component of $(g)_{\mathrm{FDM}}$. The absolute maximum sensitivity coefficient with the analytical method may also be compared to the corresponding component with the FDM.

Normalized Inner Product (NIP) Approach

The normalized inner product of the sensitivity vectors calculated by FDM and the analytical method are defined as

$$\mathrm{NIP} = \frac{(g)_{\mathrm{Anal}} \cdot (g)_{\mathrm{FDM}}}{\|(g)_{\mathrm{Anal}}\| \, \|(g)_{\mathrm{FDM}}\|} \tag{4}$$

From a geometrical viewpoint, NIP is the cosine of the angle between two sensitivity vectors; when NIP is 1, the two vectors coincide with each other. Therefore, closeness of (1-NIP) to zero can be used as a measure of accuracy of design sensitivity analysis.

III. Example Problem

An optimal control problem is selected to demonstrate and evaluate the three verification procedures. The program IDESIGN[3] and the dynamic response optimization software OCP[4] are used in the numerical evaluation of FDM and analytical gradients. Both the adjoint variable method (AVM) and the direct differentiation method (DDM) of design sensitivity analysis are used.[2,4] The optimal control problem[4,5] is to find control function $u(t)$ to minimize the performance index

$$\psi = \int_0^1 [z^2(t) + u^2(t)] \, \mathrm{d}t \tag{5}$$

subject to the state equation $\dot{z} = z^2 - u$ with initial condition $z(0) = 1$, the terminal condition treated as an equality constraint $h \equiv z(1) - 1 = 0$, and dynamic state variable constraint $\phi \equiv -z(t) + 0.9 \le 0$, $0 \le t \le 1$.

The initial control function in the time interval [0, 1] is set to 1, which implies that the parameterized design variable $b_i = 1$ for each $i = 1$, NU. The small number of grid points for the control function ($NU = 5$), which also gives a small number of design variables, is used to present sensitivity coefficients in detail. The number of time grid points for the state variable is 101. The piecewise cubic spline interpolation scheme is used for state and control variables in numerical integrations. The perturbation of design variable δ is set to 0.01% after some numerical experimentation. This is usually necessary to ensure accuracy of the derivatives by the FDM. The DDEABM[4] program with absolute error of 1.E − 7 and relative error of 3.E − 7 is employed to solve first-order differential equations. Gaussian quadrature formula with 128 Gaussian points is used to carry out numerical integration during sensitivity calculations.

Comparing sensitivity coefficients for the performance index ψ and terminal constraint h, given in Table 1, we see that agreement between DDM and FDM is very good. Since a small

Table 1 Sensitivity coefficients for performance index and terminal constraint, five design variables

Performance index, ψ	FDM	DDM, %	AVM, %
$d\psi/db_1$	−6.9477E−01	−6.9486E−01 (100.0)	−7.1019E−01 (102.2)
$d\psi/db_2$	−1.4064E−01	−1.4054E−01 (99.9)	−1.2523E−01 (89.0)
$d\psi/db_3$	2.0633E−01	2.0638E−01 (100.0)	2.0930E−01 (101.4)
$d\psi/db_4$	4.3448E−01	4.3450E−01 (100.0)	4.3159E−01 (99.3)
$d\psi/db_5$	−3.0815E−07	−4.5974E−13 (0.0)	0.0000E+00 (0.0)
$\Delta\psi$	−1.9413E−05	—	—
ψ'	—	−1.9452E−05 (100.2)	−1.9453E−05 (100.2)
(1-NIP)	—	1.0080E−08	2.7152E−04
RE	—	1.2953E−04	2.2194E−02

Terminal constraint, h	FDM	DDM, %	AVM, %
dh/db_1	−1.4536E+00	−1.4537E+00 (100.0)	−1.4734E+00 (101.3)
dh/db_2	−8.8181E−01	−8.8170E−01 (99.9)	−8.6198E−01 (97.7)
dh/db_3	−5.3485E−01	−5.3478E−01 (99.9)	−5.2744E−01 (98.6)
dh/db_4	−3.2413E−01	−3.2436E−01 (100.0)	−3.3166E−01 (102.3)
dh/db_5	−3.9062E−05	−5.8282E−11 (0.0)	0.0000E+00 (0.0)
Δh	−3.1942E−04	—	—
h'	—	−3.1945E−04 (100.0)	−3.1945E−04 (100.0)
(1-NIP)	—	1.2066E−08	1.3141E−04
RE	—	6.8795E−05	1.3572E−04

Note: (*xxx*) shows percent of FDM.

Table 2 Comparison of sensitivity coefficients, 51 design variables

		DDM		AVM	
	Time	(1-NIP)	RE	(1-NIP)	RE
ψ		3.9738E−04	−9.1974E−02	3.6382E−02	−9.6302E−04
h		1.4343E−04	−3.5796E−03	1.7100E−02	−6.2285E−03
	0.0	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
	0.1	3.7862E−04	3.6378E−03	7.3483E−04	5.8031E−03
	0.2	2.4603E−04	3.6166E−03	8.8467E−04	3.2408E−02
	0.3	1.9747E−04	3.6288E−03	2.1142E−03	3.2832E−02
	0.4	1.7389E−04	3.6127E−03	2.7782E−03	7.6294E−02
ϕ	0.5	1.6111E−04	3.6001E−03	7.8610E−03	1.0275E−02
	0.6	1.5327E−04	3.5848E−03	1.3622E−02	1.5102E−01
	0.8	1.4638E−04	3.5672E−03	8.3019E−03	8.5452E−02
	0.9	1.4464E−04	3.5852E−03	1.0595E−02	3.2966E−02
	1.0	1.4343E−04	3.5796E−03	1.7200E−02	6.2285E−03

number of control grid points is used, difference between the second sensitivity components for the performance index ψ calculated with AVM and FDM is 11%. Note that even though the DDM and AVM have different sensitivity coefficients, the first variations agree almost 100%. Comparing (1-NIP) and RE for performance index ψ, we observe that the order of magnitudes is different. For example, the order of (1-NIP) for DDM is -8, and AVM is -4, and the order of RE for DDM is -4, and AVM is -2. This also is true for the sensitivity coefficients of the terminal constraint h. This indicates that DDM gives more accurate gradients than AVM for this application, as may be observed from the data given in Table 1.

A large number of control grid points ($NU = 51$) is also used to compare the sensitivity coefficients for all the constraints. Other data are the same as in the preceding case. Table 2 contains comparison of sensitivity coefficients for ψ, h, and ϕ. Because of the large amount of data, component by component comparison of the sensitivity coefficients is not included in the table. Note that $d\psi/d\boldsymbol{b}$ and $dh/d\boldsymbol{b}$ have 51 components, and $d\phi/d\boldsymbol{b}$ is a matrix of gradients of dimension 51×101 because there are 101 time grid points. Also, Table 2 does not contain comparisons of first variations because they were all very close to 100 with both the AVM and DDM. Also for ϕ, comparisons for only 11 out of 101 time grid points are shown in the table; the remaining points had similar results. It may be observed from the data in Table 2 that the accuracy of the analytical methods is good. However, DDM is more accurate than AVM for this example. This agrees with the results previously reported.[4] In addition, DDM is easier to implement in a computer program.

IV. Discussion and Conclusions

In this paper, the problem of verifying design sensitivity analysis when implemented into a computer program is addressed. This verification is quite important because incorrect gradients can cause the optimization process to fail or converge to nonoptimum points. Three verification methods are presented: the first variation, normalized inner product, and relative error in the largest component. It is observed that the first variation method is considerably less expensive computationally than the other two methods. In the method, all design variables are changed in $\Delta \boldsymbol{b}$ to evaluate the first variation by finite differences. However, in the other two methods, design variables must be changed one by one and functions re-evaluated to calculate the gradient vector by the finite-difference method.

The first variation method can measure only the global accuracy of sensitivity calculations. It is shown that even if global accuracy is good, there can be large errors in components of the sensitivity vector. Therefore, this method is not suitable for verifying design sensitivity analysis and is not recommended for general usage.

It is clear that if components of the sensitivity vector are good, then global accuracy is automatically good. Therefore, the best approach is to verify each component of the sensitivity vector. However, when there is a large difference in the magnitude of the calculated gradient components, the method can indicate large errors even if sensitivity calculations are correct. So, the approach should be used with caution and should be augmented with the normalized inner product or the relative error criterion. This procedure of verifying sensitivity calculations is recommended for general usage. However, since the procedure can be computationally expensive, it is recommended to first verify the sensitivity calculations on several small-scale problems before testing the program on large-scale applications.

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Mass Matrix Modification Using Element Correction Method

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Introduction

IN a dynamic analysis, the mode shapes obtained from vibration tests are generally not orthogonal to the analytical mass matrix. Several methods have been published which assume that the analytical mass matrix is correct and thereby modify the measured modes to achieve orthogonality.[1-4] An alternate approach is to assume that the measured mode shapes are exact and thus the analytical mass matrix is modified.[5-8]

The reason that the mass matrix correction combined with the Lagrange multiplier method has merit is that the resulting analytical model predicts the results obtained from experiments exactly. Also, this method is simpler and requires less computer time than other approaches. However, this method has limitations: 1) the banded character of the mass matrix is destroyed after the modification and the modified mass matrix does not represent the mass distribution of an actual structure, and 2) the total weight of the system is changed from the test value after the modification.

In this Note, an analytical mass matrix modification method via element correction is proposed based on an incomplete set of modal test data. This approach has the capability to change only those elements that require modification and, therefore, preserves a banded or a full mass matrix. Furthermore, the total weight constraint is also enforced in the analysis. It is felt that the present method is a viable technique for improving an analytical model based on an incomplete set of test data.

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